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Rotating soliton solution in Einstein-Maxwell-dilaton-axion gravity

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We demonstrate the possibility of the application of the inverse scattering problem technique to a chiral string system with the nontrivial group condition. We consider the Einstein-Maxwell theory with dilaton and axion fields and construct the massive rotating solution with nontrivial fields characteristics by use of the Belinskiĭ-Zakharov *L-A* pair.

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I. INTRODUCTION

Recently much attention has been given to the study of gravity models appearing in the low energy limit of superstring theory [1]. Some of them possess the chiral matrix representation in the stationary case that allows us to apply the different mathematical methods for the construction of exact solutions.

The model under consideration is the Einstein-Maxwell theory with dilaton and axion fields (EMDA). It appears in the framework of heterotic string theory after the omission of a part of the fields arising during extra dimension compactification. As has been established earlier, the three-dimensional chiral matrix of this theory belongs to the $Sp(4,R)/U(2)$ coset representation [2].

The EMDA theory is well investigated and a number of exact solutions of this theory is constructed [2–4]. Therefore it would be interesting also to apply the inverse scattering problem technique (IST) [2,5–7] to this model for the construction of exact solutions.

In this paper we continue to consider the IST application to chiral theories with matrix dimensions greater than two. Starting from a trivial group model [8], we generalize the result to the case of the symplectic group. The solution obtained depends on a number of constants, and both the symmetry and group conditions reduce to the restrictions on these constants. This way differs from the one proposed in [7]; for the real group the group requirement in the case of an arbitrary value of the spectral complex parameter seems to be not obvious.

By use of the Belinskiĭ-Zakharov *L-A* pair for the two-soliton case we construct the nontrivial axially symmetric metric and field configuration from the trivial one. This configuration corresponds to a rotating massive source with all physical charges, possessing a NUT (Newman-Unti-Tamburino) parameter. Since the scheme proposed does not depend on a matrix dimension of the theory, it is possible to apply it to a soliton solution construction for the string gravity models with arbitrary matrix dimension.

II. MODEL UNDER CONSIDERATION

The EMDA theory taking into account dilaton, axion, and Maxwell fields coupled to gravity is described by the action

$$S^{(4)}=\int d^4x|g|^{1/2}\left(-R^{(4)}+2\partial\phi^2+\frac{1}{2}e^{4\phi}\partial\kappa^2-e^{-2\phi}F^2-\kappa F\tilde{F}\right). \tag{2.1}$$

Here $R=R^{\mu\nu}_{..\mu\nu}$ is the Ricci scalar $(R^{\mu}_{\mu\lambda\sigma}=\partial_\lambda\Gamma^{\mu}_{\mu\phi\dots})$ of the four-metric $g_{\mu\nu}$,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\qquad \tilde{F}^{\mu\nu}=\frac{1}{2}E^{\mu\nu\lambda\sigma}F_{\lambda\sigma}, \tag{2.2}$$

ϕ is the scalar dilaton field, and the axion is written in the form of pseudoscalar field κ.

Below we consider the stationary case when the metric and matter fields are time independent. The four-dimensional line element can be parametrize according to [9]

$$ds^2=f(dt-\omega_i dx^i)^2-f^{-1}h_{ij}dx^i dx^j, \tag{2.3}$$

where $i=1,2,3$. As it has been shown in [3], in this case part of the Euler-Lagrange equations can be used for the transition from the spatial components of vector potential A_i and metric functions ω_i to the magnetic u and rotational $\tilde{\chi}$ potentials, respectively. The new and old variables are connected by the differential relations

$$\nabla u=fe^{-2\phi}(\sqrt{2}\nabla\times\vec{A}+\nabla v\times\vec{\omega})+\kappa\nabla v, \tag{2.4}$$

$$\nabla\tilde{\chi}=u\nabla v-v\nabla u-f^2\nabla\times\vec{\omega} \tag{2.5}$$

(here $v=\sqrt{2}A_0$, and three-dimensional operator ∇ is connected with the metric h_{ij}). Then the resulting three-dimensional model can be described by the action [2]

$$S=\int d^3xh^{1/2}\left(-R+\frac{1}{4}\mathrm{Tr}(J^M)^2\right), \tag{2.6}$$

where $J^M=\nabla MM^{-1}$, and R is the curvature scalar constructed on the three-metric h_{ij}. The symmetric matrix M has the form

$$M=\begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & P+QP^{-1}Q \end{pmatrix}, \tag{2.7}$$

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where the symmetric 2×2 matrices P and Q are

$$P=\begin{pmatrix} f-v^2e^{-2\phi} & -ve^{-2\phi} \\ -ve^{-2\phi} & -e^{-2\phi} \end{pmatrix}, \tag{2.8}$$

$$Q=\begin{pmatrix} -\tilde{\chi}+vw & w \\ w & -\kappa \end{pmatrix}, \tag{2.9}$$

and $w=u-\kappa v$.

The matrix M belongs to $Sp(4,R)/U(2)$ coset representation [2] and, hence, satisfies the symplectic and symmetric properties

$$M^TLM=L, \text{ and } M^T=M, \tag{2.10}$$

where

$$L=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Next, we will consider the stationary axisymmetric field configurations. The metric and matter fields depend only on two space coordinates, and the three-dimensional line element can be taken in the Lewis-Papapetrou form:

$$ds_3^2=h_{ij}dx^idx^j=e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\varphi^2. \tag{2.11}$$

The action of the system becomes

$${}^2S=\frac{1}{4}\int d\rho dz\rho\,\mathrm{Tr}(J^M)^2, \tag{2.12}$$

and the chiral matrix Euler-Lagrange equation reads

$$\nabla(\rho J^M)=0. \tag{2.13}$$

The Einstein equations become the relations defining the metric function γ:

$$\gamma_{,z}=\frac{\rho}{4}\mathrm{Tr}[J^M_\rho J^M_z],$$
$$\gamma_{,\rho}=\frac{\rho}{8}\mathrm{Tr}[(J^M_\rho)^2-(J^M_z)^2]. \tag{2.14}$$

In these relations all variables depend on two coordinates ρ and z, and the operator ∇ is connected with the flat two-metric δ_{ab}.

In the stationary axisymmetric case the system under consideration is completely described by Eqs. (2.13),(2.14). Now let us consider this system with nontrivial group condition (2.10) and apply the inverse scattering problem method for construction of soliton configuration from the trivial fields and space-time metric.

III. INVERSE SCATTERING PROBLEM METHOD

By use of the Belinskiĭ and Zakharov L-A pair [5] we would like to employ the modified scheme proposed in [8] for a case of matrix model with nontrivial group condition. So, we consider the Einstein-Maxwell system with dilaton and axion fields, possessing the $Sp(4,R)$ isometry group, locally isomorphic to $SO(2,3)$ one. This gives the reason to believe that the scheme presented here may be applicable to the case of arbitrary orthogonal group string theory.

Let us describe the main aspects of the scheme used. The axially symmetric motion equations (2.13) read

$$\nabla(\rho J^M)=0, \text{ where } J^M=\nabla M\,M^{-1}, \tag{3.1}$$

$\nabla_i=\partial_i$, $i=\rho,z$. The integration of matrix equation (3.1) is associated with the L-A pair [5]:

$$D_1\psi=\frac{\rho J^{\mathcal{M}}{}_z-\lambda J^{\mathcal{M}}{}_\rho}{\lambda^2+\rho^2}\psi, \quad D_2\psi=\frac{\rho J^{\mathcal{M}}{}_\rho+\lambda J^{\mathcal{M}}{}_z}{\lambda^2+\rho^2}\psi, \tag{3.2}$$

where $J^{\mathcal{M}}=\rho J^M$ and the differential operators D_1 and D_2 are

$$D_1=\partial_z-\frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda, \quad D_2=\partial_\rho+\frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda; \tag{3.3}$$

λ is the spectral complex parameter and the function $\psi=\psi(\lambda,\rho,z)$. The solution of Eq. (3.1) for the matrix M is represented as

$$M(\rho,z)=\psi(0,\rho,z). \tag{3.4}$$

The function ψ can be obtained in the form

$$\psi=\chi\psi_0, \tag{3.5}$$

where ψ_0 is some known solution of the system (3.2),(3.3). The equations for χ are

$$D_1\chi=\frac{\rho J^{\mathcal{M}}{}_z-\lambda J^{\mathcal{M}}{}_\rho}{\lambda^2+\rho^2}\chi-\chi\frac{\rho(J^{\mathcal{M}}{}_z)_0-\lambda(J^{\mathcal{M}}{}_\rho)_0}{\lambda^2+\rho^2}, \tag{3.6}$$

$$D_2\chi=\frac{\rho J^{\mathcal{M}}{}_\rho+\lambda J^{\mathcal{M}}{}_z}{\lambda^2+\rho^2}\chi-\chi\frac{\rho(J^{\mathcal{M}}{}_\rho)_0+\lambda(J^{\mathcal{M}}{}_z)_0}{\lambda^2+\rho^2}. \tag{3.7}$$

It is necessary that the resulting matrix solution be real and symmetric, as well as it must satisfy the group requirement (2.10). The former condition is ensured because we will consider only the case of a real matrix χ, but the other ones can be attained after the solution will be obtained.

The soliton solutions for the matrix M correspond to the pole divergence in the spectral parameter complex plane for the matrices χ and χ^{-1}. For the simple poles, these matrices may be represented as

$$\chi=I+\sum_{k=1}^{N}\frac{R_k}{\lambda-\mu_k},\quad \chi^{-1}=I+\sum_{k=1}^{N}\frac{S_k}{\lambda-\nu_k},\tag{3.8}$$

where the pole trajectories for each pole k are determined by

$$\mu_k(\rho,z)=w_{(\mu)}-z\pm[(w_{(\mu)}-z)^2+\rho^2]^{1/2},\quad w_{(\mu)}=\text{const}\tag{3.9}$$

for $\mu_k(\rho,z)$ and the same equation for $\nu_k(\rho,z)$ with the constant $w_{(\nu)}$. From the obvious relation $\chi\chi^{-1}=I$ (in the poles μ_k and ν_k) it follows that

$$R_k\chi^{-1}(\mu_k)=S_k\chi(\nu_k)=0.\tag{3.10}$$

This demonstrates that the matrices R_k and S_k are degenerate and may be presented in the form

$$(R_k)_{ab}=n_a^k m_b^k,\quad (S_k)_{ab}=p_a^k q_b^k.\tag{3.11}$$

The substitution of Eqs. (3.8) and (3.11) in Eq. (3.10) gives

$$n_a^k=\sum_{l=1}^{N}p_a^l\Gamma_{kl}^{-1},\ \ q_a^k=-\sum_{l=1}^{N}m_a^l\Gamma_{kl}^{-1},\quad \text{where}\ \ \Gamma_{kl}=\frac{\Sigma_c p_c^k m_c^l}{\mu_l-\nu_k},\tag{3.12}$$

and one can see that [6]

$$m_a^k=[\psi_0^{-1}(\mu_k,\rho,z)]_{ca}m_{c0}^k,\quad p_a^k=[\psi_0(\nu_k,\rho,z)]_{ac}p_{c0}^k,\tag{3.13}$$

where m_{c0}^k and p_{c0}^k are the arbitrary constants.

Since the matrix $M(\rho,z)$ belongs to $Sp(4,R)$, it must be unimodular. As has been demonstrated in [8] for the two-solitons configuration, it is important that

$$\mu_1\mu_2=\nu_1\nu_2.\tag{3.14}$$

As before, we consider the two-solitons case, and one may generalize this to the case of 2N-solitons.

The resulting solution for the matrix M is unimodular, however it does not satisfy the requirements (2.10). The $Sp(4,R)/U(2)$ coset representation of this solution may be attained by the suitable choice of the arbitrary constants in Eq. (3.13). Hence, as has been mentioned above, we satisfy the group condition after the formal matrix solution is obtained. This will be demonstrated in the next section.

IV. EXACT SOLITON SOLUTION

Now let us apply the inverse scattering problem method to the construction of the stationary axial-symmetric two-soliton solution for the Einstein-Maxwell system with dilaton and axion fields.

It is natural to determine the asymptotic values of the fields as

$$f_\infty=1,\ \ \tilde{\chi}_\infty=u_\infty=v_\infty=\phi_\infty=\kappa_\infty=0,\tag{4.1}$$

and to put the initial value of matrix $M_0=M_\infty$:

$$M_0=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},\quad \text{where}\ \ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\tag{4.2}$$

Then, we construct the solution in the Boyer-Lindquist coordinates:

$$\rho=[(r-m)^2-\sigma^2]^{1/2}\sin\theta,\quad z-z_1=(r-m)\cos\theta;\tag{4.3}$$

the new constants $\sigma=1/2(w_{(\mu)}-w_{(\nu)})$ and $z_1=1/2(w_{(\mu)}+w_{(\nu)})$ [see Eq. (3.9)]. Following the IST scheme we obtain the expressions for the pole trajectories:

$$\mu_1=2\sin^2\frac{\theta}{2}[r-m+\sigma],\ \ \mu_2=-2\cos^2\frac{\theta}{2}[r-m-\sigma],$$
$$\nu_1=-2\cos^2\frac{\theta}{2}[r-m+\sigma],\ \ \nu_2=2\sin^2\frac{\theta}{2}[r-m-\sigma],\tag{4.4}$$

that satisfy the condition (3.14).

The resulting matrix solution of Eq. (3.1) for M is only unimodular and the main aim now is to provide both symmetry and group requirement. To do this one may note that as $\psi_0{}^{-1}(\mu_k,\rho,z)=\psi_0(\nu_k,\rho,z)=M_0$, the vectors p_a^k and m_a^k become constants [see Eq. (3.13)]. This makes the choice of some Anzätze easier for these constants.

At first, we consider four columns p_a^k and m_a^k, $k=1,2$, $a=0,1,2,3$ and put

$$p^1=\Lambda p^2,\quad m^1=-\Lambda m^2,\quad \text{where}\ \ \Lambda=\begin{pmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}.\tag{4.5}$$

These relations leave eight independent parameters, which can be combined into two matrices $p=\|p_a^k\|$ and $\overline{m}=\|m_a^k\|$, $k=1,2$, $a=1,2$. Then, the additional restrictions, that provide Eq. (2.10) read

$$\text{Tr}\,p^T\sigma_1\overline{m}=0,\quad \text{Tr}\sigma_2\,p^T\sigma_1\overline{m}=0,$$
$$\text{where}\ \ \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.\tag{4.6}$$

Hence, one can see that the coset representation may be realized on a matrix solution obtained after imposing some additional conditions (4.5),(4.6) on the constants in this solution. As has been mentioned above, this approach differs from the one proposed in [7]; the authors consider the group condition on the matrix solution with arbitrary value of complex parameter λ, but for the case of the real group it seems to be complicated.

Let us now present the expressions obtained for the metric and matter fields. One may naturally determine the physical charges of the system. So, by entering the mass m, the parameter NUT b, the electric Q_e, magnetic Q_m, dilaton D

and axion K charges, one may write the main parts of the asymptotic decomposition of the metric and matter fields at $r\to\infty$:

$$f\to 1-\frac{2m}{r},\quad \tilde{\chi}\to\frac{2b}{r},$$

$$v\to\frac{\sqrt{2}Q_e}{r},\quad u\to\frac{\sqrt{2}Q_m}{r},$$

$$\phi\to\frac{D}{r},\quad \kappa\to\frac{2K}{r}. \tag{4.7}$$

The above charges can be expressed in terms of the vector components p_a^k and m_a^k as follows:

$$m=\sigma(\tilde{p}_2^2m_2^1-\tilde{p}_2^1m_2^2),\quad b=-\sigma(\tilde{p}_2^2m_2^2+\tilde{p}_2^1m_2^1), \tag{4.8}$$

$$Q_e=\sqrt{2}\sigma(\tilde{p}_2^2m_1^2+\tilde{p}_2^1m_1^1),\quad Q_m=\sqrt{2}\sigma(\tilde{p}_1^2m_2^1-\tilde{p}_1^1m_2^2), \tag{4.9}$$

$$D=\sigma(\tilde{p}_1^1m_1^2-\tilde{p}_1^2m_1^1),\quad K=-\sigma(\tilde{p}_1^1m_1^1+\tilde{p}_1^2m_1^2), \tag{4.10}$$

where $\tilde{p}_a^k=p_a^k/\mathrm{Tr}\bar{m}\sigma_1p^T$. In addition, we determine the Kerr parameter a as

$$a=-\sigma\,\mathrm{Tr}\,\bar{m}\sigma_3\tilde{p}^T, \tag{4.11}$$

and one can see that

$$m^2+b^2+D^2+K^2-Q_e{}^2-Q_m{}^2-a^2=\sigma^2. \tag{4.12}$$

If one introduces the notation

$$\Delta=(r-m)^2-\sigma^2,\quad \delta^2=r^2+(b-a\cos\theta)^2-D^2-K^2, \tag{4.13}$$

the four-dimensional line element can be presented in the form

$$ds^2=f(dt-\omega_\varphi d\varphi)^2-f^{-1}ds_3^2, \tag{4.14}$$

where

$$f=\frac{\Delta-a^2\sin^2\theta}{\delta^2}. \tag{4.15}$$

The metric function ω_φ is determined as

$$\omega_\varphi=\frac{2}{\Delta-a^2\sin^2\theta}\Big[b\cos\theta\Delta-a\sin^2\theta \times\left(mr+b^2-\frac{1}{2}(Q_e{}^2+Q_m{}^2)\right)\Big], \tag{4.16}$$

and one can see that the constants a and b are actually the rotation and the NUT parameter, respectively.

Next, the three-dimensional metric has the form

$$ds_3^2=\frac{\Delta-a^2\sin^2\theta}{\Delta}dr^2+(\Delta-a^2\sin^2\theta)d\theta^2+\Delta\sin^2\theta d\varphi^2 \tag{4.17}$$

that coincides with the Kerr three-metric.

The dilaton function is

$$e^{2\phi}=\frac{(r+D)^2-(b-a\cos\theta-K)^2}{\delta^2}, \tag{4.18}$$

whereas the pseudoscalar axion field read

$$\kappa=\frac{2[Kr+D(b-a\cos\theta)]}{(r+D)^2-(b-a\cos\theta-K)^2}. \tag{4.19}$$

For the electric and magnetic potentials one obtains

$$v=\frac{\sqrt{2}[Q_er-Q_m(b-a\cos\theta)+DQ_e+KQ_m]}{\delta^2} \tag{4.20}$$

and

$$u=\frac{\sqrt{2}[Q_mr+Q_e(b-a\cos\theta)+DQ_m-KQ_e]}{\delta^2}. \tag{4.21}$$

Thus the above expressions describe the stationary axially symmetric massive configuration possessing the all possible charges and NUT parameter. The latter characteristics does not permit to interpret this source properly as a black hole, in spite of a horizon presence.

This solution seems to be original. Although the expressions for the some metric components or fields look like the ones obtained earlier [4,10,11], one may see that in the solution presented all physical charges of the system are independent to a certain degree. So, the conditions (4.5) and (4.6) leave six independent parameters p_a^k and m_a^k, that occur in the expressions (4.8)–(4.11) for the physical charges of the system, and as a result of this the condition for these charges and angular momentum is Eq. (4.11). The solution differs from the one obtained in [11]; we have not free parameters corresponding to dilaton and axion values at infinity because of asymptotic behavior (4.7), and the charges of the system do not always satisfy the constraint $Q_mm=Q_eb$.

By some choice of constants p_a^k and m_a^k one may obtain the massless configuration with all other charges, that requires the adequate interpretation. More specifically, the different choice of above constants allow to describe some of models known, as a electric rotational configuration with mass and dilation charge, as well as the dilaton-axion gravity without vector fields. Certainly, the solution presented contains Kerr-NUT metric; however, it does not describe the extremal sources because of $\sigma\neq 0$.

Hence, the solution obtained is a nonextremal rotating black-hole *type* [4] source, possessing all possible physical charges of the EMDA system.

V. DISCUSSION

In this paper we continue to investigate the IST application to chiral matrix models with dimension greater than two. The Einstein-Maxwell gravity with dilaton and axion fields is one of such model. In the framework of this theory we obtain the soliton solution from the trivial metric and fields configuration by use of Belinskiĭ and Zakharov *L-A* pair [5] and the result of [8]. This solution describes the rotating charged object with NUT parameter, that includes, for example, Kerr-NUT source and an electrically charged dilatonic black hole.

It would be interesting to further develop the formalism under consideration and to connect it with the Geroch group [12] construction for the chiral matrix models.

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